Two-Point Boundary Value Problems Associated With First Order Fuzzy Differential Equations–Existence and Uniqueness

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Abstract:

Existence and Uniqueness of solutions of Two-Point boundary value problems associated with first order fuzzy differential system are established. Differential inclusion approach is used as a tool to obtain existence and uniqueness.

Key words:

Fuzzy sets and systems, Fuzzy rule, Green's Matrix,Dynamicalsystems,Lipschitz condition

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1. Introduction:

In modeling real systems one can be frequently confronted with a system of first order matrix differential equation. Fuzzy boundary value problems constitute a very interesting and important class as it includes differential inclusions. The theory of fuzzy sets, fuzzy valued functions and necessary calculus of fuzzy functions have been discussed in the recent monograph by Lakshmikantham and Mohapatra [8]. Existence and Uniqueness of initial value problems associated with first order fuzzy differential equations.

$$y' = f(t, y) \tag{1.1}$$

was obtained by Kaleva [1] under the usual assumptions. Further Nieto [9] obtained a version of the Peano's existence theorem for fuzzy differential equations associated with (1.1) assuming that f is continuous and bounded for all $y^{\alpha} \in R$. In the year 2001, Lakshmikantam, Murty and Turner [7] obtained existence and uniqueness criteria for second order fuzzy differential equations with the help of Greens function and an application of Banach fixed point theorem. The existence and uniqueness of solutions to general first order fuzzy differential equations satisfies general fuzzy boundary conditions.UsingDifferential inclusions is a significant topic that needs investigation. In this paper, we consider the general first order fuzzy differential equation

$$(y^{\alpha})' = A(t)y^{(\alpha)} + f(t) \qquad (2.1)$$
$$My^{\alpha}(a) + Ny^{\alpha}(\beta) = \gamma$$

where A is an $(n \times n)$ continuous matrix, y^{α} is a column matrix with components $(y_1^{\alpha}, y_2^{\alpha}, \dots, y_n^{\alpha})$
and$M$ and N are constant matrices of order $(n \times n)$ and all scalars are assumed to real and $\alpha \in [0,1]$

Recently Kasiviswanadhet. al. [14] obtained existence and uniqueness of solutions to first order matrix system satisfying boundary conditions at three points and extended their ideas into fuzzy linear system of differential equations using differential inclusions. In this paper our main interest is to establish existence and uniqueness solutions to first order fuzzy differential equations satisfying the most general boundary conditions for each $\alpha \in [0,1]$. To establish our main result, we very much make use of the results established for linear systems by Kasiviswanadh, et. al [11] and Yan Wu, Divya et.al [12]. For results on control systems we refer to [13]. The metricsthat weuse in this paper are taken from [6]. Further results on fuzzy sets and systems are taken from [3,4,5,10]. In the year 2020,KasiViswanadh V. Kanuri established Ψ-bounded solutions of linear systems on Timescales[15]. This is a notable contribution to the theory of differential equations.

2. Preliminaries:

In this section,we present notions and preliminary results on fuzzy sets and systems used in our paper for later discussions.

Definition 2.1: A sequence of functions $\Phi_n^{\alpha}: R^+ \longrightarrow R^k$ is said to be Ψ^{α}-bounded on R^+ if $\Psi^{\alpha}(t)\Phi_n^{\alpha}(t)$ are bounded on $R^+$for every $\alpha \in [0,1]$.

Let A be a continuous (nxn) associated with first order fuzzy matrix differential system

$$(x^{\alpha})' = A(t)x^{\alpha} \quad (2.2)$$

Let $\Phi^{\alpha}(t)$be a fundamental matrix of (2.2) for each $\alpha \in [0,1]$ satisfying $\Phi^{\alpha}(0) = I_k$ (a unit matrix of order k) and let X_1 be the subspace ofR^k consisting of all vectors which are values of Ψ^{α}-bounded solutions of (2.2) for $t = 0$ and X_2 be an arbitrary fixed subspace of R^k supplementary to X_1 and let P_1 be the projection of R^k onto $X_1(P_1^2 = P_1)$ and $P_2 = I - P_1$ be the orthogonal projection onto X_2.

Definition 2.2: Let X be a non-empty set. A fuzzy set A in X is characterized by its membership function $A: X \longrightarrow [0,1]$and $A(x)$ for $x \in A$is interpreted as the degree of the membership of the element x in the fuzzy set A.
The value of 'zero' is interpreted as the complete non-membership and the value of 'one' interpreted as complete membership and the values in between 0 and 1 are interpreted as degree of membership.

As an example consider

$$A(x) = \exp\{-\beta(x-1)^2\},$$

where$\beta > 0$ is a membership function of the fuzzy set of real numbers closed to one.

Let $P_k(R^n)$ denote the family of all non-empty compact convex subset of R^n. Define the addition and scalar multiplication in $P_k(R^n)$ as

$$\alpha(A+B) = \alpha A + \alpha B\ .$$

$$\alpha(\beta A) = \alpha\beta A, \quad 1 \cdot A = A.$$

and if $\alpha, \beta \geq 0$ then $(\alpha+\beta)A = \alpha A + \beta A$ and $(\alpha\beta)A = \alpha(\beta A)$

The distance between A and B is defined as the Hausdorff metric

$$d(A,B) = Inf\{\epsilon: A \subset N(B,\epsilon), B \subset N(A,\epsilon)\}$$

where$N(A,\epsilon) = \{x \ \in R^n : \parallel x - y \parallel < \epsilon \ for\ some\ y \in A\}$

Let $I = [a,b] \subset R$ be a compact subinterval and we denote

$$E^n = \{u: R^n \to [0,1] \text{ / such that } u \text{ satisfies}\}$$

(i) u is normal there exists an $x_0 \in R^n$ such that $u(x_0) = 1$
(ii) u is fuzzy convex i.e., for any $x, y \in R^n$ and $0 \leq \lambda \leq 1$

$$u(\lambda x + (1-\lambda)y \geq \min\{u(x), u(y)\}$$

(iii) u is upper semi continuous and

(iv) $[u]^0 = d\,\{x \in R^n \,/\, u(x) > 0\}$is compact.

It may be noted that, by fundamental sequence on a complete normed linear space, X, we mean a sequence in X whose span Y is dense in X. In other words, every element in X can be approximated by an element in the span. That is, given $x \in X$, we can find a $y \in Y$ such that $\parallel x - y \parallel < \epsilon$.

For any $0 \leq \alpha \leq 1$, the α-level set is denoted by

$$u^\alpha = \{x \in R^n / u(x) \geq \alpha\}$$

Define $D : E^k X E^k \to [0, \infty)$ by

$$D(u, v) = \sup\{d[u]^\alpha, d[v]^\alpha\}$$

$$= \sup\{(d[u]^\alpha, [v]^\alpha)\}\, \alpha \in [0,1]$$

where d is the Hausdorff metric. It may be noted that (E^k, D) is a Banach space. D has a linear structure.

For any $u, v, w \in E^k$, $\lambda \in R$, we have

$$D(u + w, v + w) = D(u, v)$$

$$D(\lambda u, \lambda v) = |\lambda| D(u, v)$$

Note that (E^k, D) is not a metric space. But it can be in fact imbedded isomorphically as a cone in Banach space.

Ascoli Lemma:

Let X be a Banach space and Y be any metric space. A subset $\Phi$of the space $\zeta(X, Y)$ of continuous mapping of X into Y is totally bounded in the metric of the uniform convergence if and only if Φ is equicontinuous and uniformly bounded subset Y for each $x \in X$.

Lemma:$\Phi : T \to E^k$ is a solution of the initial value problem

$$x' = f(t, x); \quad x(t_0) = x_0$$

if and only if Φ is a solution of the integral equation

$$x(t) = x_0 + \int_{t_0}^{t} f\big(s, x(s)\big) ds \text{ ,for all } t \in T$$

Definition 2.3: Let $\alpha \in [0,1]$ and $f: T \times E^k \to E^k$ be continuous. We say that, f satisfies a Lipschitz condition with the Lipschitz constant $k > 0$, if for any

$$(t, x^\alpha), (t, y^\alpha) \in T \times E^k$$

$$D\{f\big((t, x^\alpha), f(t, y^\alpha)\big\} \leq k\, D(x^\alpha, y^\alpha) \qquad (2.3)$$

We now concentrate on the fuzzy first order linear non-homogeneous system

$$(x^\alpha)' = A(t)x^\alpha + f(t'p \qquad (2.4)$$

where$A$ is a (kxk) continuous matrix $x^\alpha = \{x_1^\alpha, x_2^\alpha, \ldots, x_k^\alpha\}$. Here after, we denote

$x^\alpha = \hat{x}$and hence (2.4) can be written as

$$(\hat{x})' = A(t)\hat{x} + f(t)$$

Definition 2.4: Let $\alpha \in [0,1]$ and $f: T \times E^k \to E^k$ be continuous. We say that the fuzzy linear system

$$(\hat{x})' = f(t, \hat{x})$$

satisfies Lipschitz condition with the Lipschitz constant if for any

$$(t, x^\alpha), (t, y^\alpha) \in T \times E^k,$$

$$D\{f\big((t, x^\alpha), f(t, y^\alpha)\big\} \leq k\, D(x^\alpha, y^\alpha).$$

For $0 \leq \alpha \leq 1$, we define

$$[x]^\alpha = \{t \in R : y(t) \geq \alpha\}$$

Further

$$g[(x, \bar{x})]^\alpha = g[x^\alpha, \bar{x}^\alpha] \text{for all } x, \bar{x} \in R$$

Definition 2.5: Let $u_i^{(\alpha)}(t)$ be the α-level set of $u_i(t)$, then we define

$$u_i^{(\alpha)}(t) = \{u_1^\alpha(t), u_2^\alpha(t), \ldots, u_k^\alpha(t)\}$$

$$= \{\hat{u}_1(t), \hat{u}_2(t), \ldots, \hat{u}_k(t)\}$$

$$= \hat{u}_i(t)$$

3. Main Result:

In this section, we shall be concerned with the existence and uniqueness criteria associated with the fuzzy two-point general boundary value problem and study the properties of the Green's matrix associated with the integral equation. We assume throughout this section that, the homogeneous fuzzy boundary value problem has only the trivial solution for each $\alpha \in [0,1]$.

Definition 3.1: A map $f: [0,1] \times E^k \to E^k$ is called level wise continuous at a point $(t_0, x_0) \in [0,1] \times E^k \to E^k$ provided, for any fixed $\alpha \in [0,1]$ and arbitrary $\epsilon > 0$, there exists a $\delta(E, \alpha) > 0$ such that

$$H_\alpha((|f(t,x)|)^\alpha, (|f(t_0,x_0)|)^\alpha) < \epsilon,$$

whenever$|t - t_0| < \delta(\epsilon, \alpha)$ and $H_\alpha(|x|^\alpha, |x|^0) < \delta(t, \alpha)$ for all $t \in [0,1]$ and $x \in E^k$

Theorem: If $u_i \in R^k$, then

(i) $[u_i]^\alpha \in P_k(R^{k\times k})$ for all $0 \leq \alpha \leq 1$
(ii) $[u_i]^{\alpha_2} \subset [u_i]^{\alpha_1}$ for $0 \leq \alpha_1 \leq \alpha_2 \leq 1$
(iii) If $\{\alpha_k\}$ is a non-decreasing sequence converging to $\alpha > 0$ then $[u_i]^\alpha = \bigcap_{k\geq 1}[u_i]^{\alpha_k}$

Conversely, if $\{u^\alpha: 0 \leq \alpha \leq 1$ is a family of subsets of R^k satisfying the above conditions (i)-(iii), then there exists a $u_i \in R^k$ such that

$$[u_i]^\alpha = u^\alpha \text{for} 0 \leq \alpha \leq 1$$

and $[u_i]^0 = u_0 u^\alpha \subset u^0$ for $0 \leq \alpha \leq 1$

Theorem: Let $u \in R^k$, then

(i) $[u]^\alpha \in P_k(R^k)$ for all $\alpha \in [0,1]$
(ii) $[u]^{\alpha_2} \subset [u]^{\alpha_1}$ for all $0 \leq \alpha_1 \leq \alpha_2 \leq 1$
(iii) If $\{\alpha_k\}$ is a non-decreasing sequence converging to $\alpha > 0$, then $\bigcap_{k\geq 1}[u]^{\alpha_k}$.

Conversely, if $\{A^\alpha: 0 \leq \alpha \leq 1\}$ is a family of subsets of R^k satisfying (i)-(iii), then there exists $u \in R^k$ such that

$[u]^\alpha = A^\alpha \text{for} 0 \leq \alpha \leq 1$ and

$[u]^0 = u_0 A^\alpha \subset A^0$

Lemma: Let$[\hat{x}(t)]^{\alpha_2} \subset [\hat{x}(t)]^{\alpha_1}$ for al$0 \leq \alpha_1 \leq \alpha_2 \leq 1$l

Proof: Let $0 \leq \alpha_1 \leq \alpha_2 \leq 1$. Since $[\hat{u}]^{\alpha_2}(t) \subset [\hat{u}]^{\alpha_1}(t)$, it bounds that

$[\hat{u}(t)]^{\alpha_2} \subset [\hat{u}(t)]^{\alpha_1}$

Thus, we have the selection inclusions $S'_{\hat{u}^{\alpha_2}} \subset S'_{\hat{u}^{\alpha_1}}$ and also the following inclusions

$\hat{x}'(t) \in A(t)\hat{x}(t) + f(t)$

Then

$$\hat{x}'(t) \in A(n)\hat{x}(t) + \{f(t)\}\hat{u}^{\alpha_2}(t),\ t \in R \tag{3.1}$$

$$\hat{x}'(t) \in A(n)\hat{x}(t) + \{f(t)\}\hat{u}^{\alpha_1}(t),\ t \in R \tag{3.2}$$

and let $\hat{x}^{\alpha_2}(t)$ and $\hat{x}^{\alpha_1}(t)$be solutions (3.1) and (3.2) respectively

clearly, the solution of (3.1) satisfies

$$\hat{x}(t) \in \varphi(t)\hat{x}_0 + \int_{t_0}^{t} \varphi(t,s)\, f(s)\hat{u}^{\alpha_2}(s)ds$$

$$\subset \varphi(t)\hat{x}_0 + \int_{t_0}^{t} \varphi(t,s)\, f(s)\hat{u}^{\alpha_1}(s)ds$$

Thus $\hat{x}^{\alpha_2} \subset \hat{x}^{\alpha_1}$ and hence $\hat{x}^{\alpha_2}(t) \subset \hat{x}^{\alpha_1}(t)$ for all $t \in R$

Lemma: If $\{\alpha_k\}$ be a non-decreasing sequence converging to $\alpha > 0$, then

$\hat{x}^{\alpha}(t) = \bigcap_{k\geq 1} \hat{x}^{\alpha_k}(t)$

Proof: Let $\hat{u}^{\alpha_k}(t) = (\hat{u}_1^{\alpha_k}(t), \hat{u}_2^{\alpha_k}(t), \dots, \hat{u}_k^{\alpha_k}(t))^T$ (3.3)

and$\hat{u}^{\alpha}(t) = (u_1^{\alpha}(t), u_2^{\alpha}(t), \dots, u_k^{\alpha}(t))^T$ (3.4)

Let $\hat{x}^{\alpha_k}(t)$ and $\hat{x}^{\alpha}(t)$ be the solution sets of (3.3) and (3.4) respectively. Since $u_i(t)$ is a fuzzy set and

$$u_i^{\alpha}(t) = \bigcap_{k\geq 1} u_i^{\alpha_k}(t)$$

and hence

$$S'_{\hat{u}^{\alpha_k(t)}} = \bigcap_{k\geq 1} S_{u^{\alpha}(t)}$$

Therefore

$$\hat{x}'(t) \in A(n)\hat{x}(t) + f(t)\hat{u}^{\alpha} \subset A(n)\hat{x}(t) + f_n\hat{u}^{\alpha_k}(t)$$

Thus

$$\hat{x}^{\alpha} \subset \hat{x}^{\alpha_k},\ k = 1,2,3, \dots.$$

Similarly, we can prove that

$$\bigcap_{k\geq 1} \hat{x}^{\alpha_k} \subset \hat{x}^{\alpha}$$

and hence

$$\hat{x}^{\alpha}(t) = \bigcap_{k\geq 1} \hat{x}^{\alpha_k}(t)$$

Theorem: The fuzzy boundary value problem

$$\hat{x}' = A(t)\hat{x} + f(t) \tag{3.5}$$

$$M\hat{x}(a) + N\hat{x}(b) = 0$$

has a unique solution and is given by

$$\hat{x}(t) \in \widehat{\Phi}(t)\hat{x}_0 + \int_{t_0}^{t} G_\alpha(t,s) f(s) ds$$

where

$$G_\alpha(t,s) \in \begin{cases} \Phi_\alpha(t) D_\alpha^{-1} M \Phi_\alpha(a) \Phi_\alpha^{-1}(s), & a \leq s \leq t \leq b \\ \Phi_\alpha(t) D_\alpha^{-1} N \Phi_\alpha(b) \Phi_\alpha^{-1}(s), & a \leq t \leq s \leq b \end{cases}$$

Proof: We first assume that the homogeneous boundary value problem has a non-trivial solutionfor each $\alpha \in [0,1]$. By variation of parameters formula any solution of non-homogeneous system is given by

$$\hat{x}(t) \in \Phi_\alpha(t)\hat{x}_0 + \Phi_\alpha(t) \int_a^t \Phi_\alpha^{-1}(s) f(s) ds \tag{3.6}$$

Substituting the general form of solution (differential inclusion) in the boundary condition matrix, we get

$$[M\Phi_\alpha(a) + N\Phi_\alpha(b)]\hat{x}_0 + N\Phi_\alpha(b) \int_a^b \Phi_\alpha^{-1}(s) f(s) ds = 0$$

Our initial assumption that the homogeneous boundary value problem for each $\alpha \in [0,1]$ has a trivial solution ensures that

$$D_\alpha = [M\Phi_\alpha(a) + N\Phi_\alpha(b)]$$

is non-singular and hence

$$\hat{x}_0 = -D_\alpha^{-1} N \Phi_\alpha(b) \int_a^b \Phi_\alpha^{-1}(s) f(s) ds$$

substituting this value of x_0 in (3.6) we get

$$\hat{x}(t) \in \Phi_\alpha(t) \left[-D_\alpha^{-1} N \Phi_\alpha(b) \int_a^b \Phi_\alpha^{-1}(s) f(s) ds \right] + \Phi_\alpha(t) \int_a^t \Phi_\alpha^{-1}(s) f(s) ds$$

$$= \int_a^b G_\alpha(t,s) f(s) ds \tag{3.7}$$

where $G_\alpha(t,s) = \begin{cases} \Phi_\alpha(t)\Phi_\alpha^{-1}(s) - D_\alpha^{-1} N \Phi_\alpha(b)\Phi^{-1}(s), \ a \le s \le t \le b \\ -\Phi_\alpha(t) D_\alpha^{-1} N \Phi_\alpha(b)\Phi^{-1}(s), \ a \le t \le s \le b \end{cases}$

The Green's matrix for the boundary value problem can further G_α can be written as

$$G_\alpha(t,s) = \begin{cases} \Phi_\alpha(t) D_\alpha^{-1} M \Phi_\alpha(a)\Phi_\alpha^{-1}(s), & a \le s \le t \le b \\ -\Phi_\alpha(t) D_\alpha^{-1} N \Phi_\alpha(b)\Phi_\alpha^{-1}(s), & a \le t \le s \le b \end{cases}$$

Theorem: Green's matrix $G_\alpha(t,s)$ in (3.7) has the following properties.

(i) For each $\alpha \in [0,1]$, G_α when considered as a function of t for a fixed s has continuous first order derivatives on $[a,s]$ and $[s,b]$ and at the point $t = s$,G has upward discontinuity of unit magnitude
i.e,$[G_\alpha(s^+,s) - G_\alpha(s^-,s)] = I_k$

(ii) G_α, for each $\alpha \in [0,1]$ is a formal solution of the fuzzy homogeneous boundary value problem and it fails to be a true solution because of the discontinuity at $t = s$

(iii) $G_\alpha(t,s)$ is a unique with properties (i) and (ii)

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